

Note on the Lunar Theory. By Professor Ernest
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I have determined the following coefficients in longitude depending on the eccentricity of the Sun. The notation used is Delaunay's. The coefficients are the parts of the form $e' f(m)$, and are accurate to the last place of decimals given.

$$\begin{aligned} V = & \dots - 659'' \cdot 2375 \sin l' + 152'' \cdot 0828 \sin (2D - l') - 21'' \cdot 5942 \sin (2D + l') \\ & + 1'' \cdot 2550 \sin (4D - l') - 0'' \cdot 1800 \sin (4D + l') \\ & + 0'' \cdot 0105 \sin (6D - l') - 0'' \cdot 0015 \sin (6D + l') \\ & + 0'' \cdot 0001 \sin (8D - l'). \end{aligned}$$

These agree well with Delaunay's values when estimate is made for the omitted portions of the series in powers of m , the greatest difference being $0'' \cdot 06$.

I hope shortly to publish the coefficients of the form $ee' f(m)$, which are amongst the least accurate of all Delaunay's coefficients owing to the slow convergence of the series in powers of m .

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On some Points connected with the Integration of the Differential Equations of the Relative Motions of Material Systems. By E. J. Stone, M.A., F.R.S., Radcliffe Observer, Oxford.

In problems of the relative motions of material points subject to their mutual gravitational attractions, when the masses are denoted by

$$1 : \nu_1 : \nu_2 : \dots : \nu_p,$$

and the corresponding coordinates, referred to the unit-mass as origin, by

$$\xi_1, \eta_1, \zeta_1; \xi_2, \eta_2, \zeta_2; \&c.,$$

the three differential equations for the motion of each mass assume the well-known forms—

$$\begin{aligned} (1) \quad & \dots \dots \dots \frac{d^2 \xi_1}{dt^2} = -f(1 + \nu_1) \frac{\xi_1}{\rho_1^3} + \int f \cdot \nu \cdot \frac{dR}{d\xi_1} \\ & \frac{d^2 \eta_1}{dt^2} = -f(1 + \nu_1) \frac{\eta_1}{\rho_1^3} + \int f \cdot \nu \cdot \frac{dR}{d\eta_1} \\ & \frac{d^2 \zeta_1}{dt^2} = -f(1 + \nu_1) \frac{\zeta_1}{\rho_1^3} + \int f \cdot \nu \cdot \frac{dR}{d\zeta_1} \\ & \&c. = \&c. \end{aligned}$$

where $\rho_1^2 = \xi_1^2 + \eta_1^2 + \zeta_1^2$; and f denotes the accelerating effect of the unit-mass at the unit-distance in the unit-time.

The symbol $\sum$ denotes summation extending over all the material points of the system. If any of the masses are so small that their accelerating effects are insensible, the corresponding symbols ν may be neglected in the summation; but if included in the mathematical work, numerical values, sensibly zero, will be found for them from a discussion of the observations when accuracy has been secured.

If found convenient the relative motions of any two material points about the unit-mass may be resolved, in the mathematical treatment, into that of the motion of their common centre of gravity about the unit-mass, and their motion about the common centre of gravity; and any other *exact* algebraical transformations of the equations may be introduced and used without modification of the general principles upon which the solution of such physical problems depend.

The $3p$ differential equations for the motion of such a system contain $(p + 1)$ constants $f, \nu_1, \nu_2, \dots, \nu_p$; and their general integrals, if determinable, will contain, in addition to these, the $6p$ constants which are required to give the necessary generality to the results, so that they may include all the possible cases of the relative motions of such a system of material points. If the adopted units of length and of time are independent of the particular system of relative motions under discussion, the whole of the $(7p + 1)$ constants will have to be found from observations of that system, and their determination will be *sufficient* for the complete solution of the physical problem. But if the times are found from the relative motions themselves, the determination of the "time" from observation will replace the determination of the two constants, to which definite values are assigned in the equations to render the variable t determinate.

If we express the coordinates under the forms

$$\xi = x + \delta x, \eta = y + \delta y, \zeta = z + \delta z,$$

and thus introduce six functions to express the position of each mass relatively to the unit-mass, we shall require for each mass, besides the three differential equations of motion, three additional equations, differential or otherwise, to render the coordinates

$$x + \delta x, y + \delta y, z + \delta z$$

determinate. When additional functions, such as $\delta x, \delta y, \delta z$, are introduced, the functions x, y, z may be called primary functions, and the functions $\delta x, \delta y, \delta z$, auxiliary functions. When the primary functions x, y, z are known, the auxiliary functions will be determinable from the differential equations.

$$(2) \quad \dots \quad \frac{d^2 \delta x}{dt^2} = -f(1 + \nu) \frac{x + \delta x}{\rho^3} - \frac{d^2 x}{dt^2} + \int f \cdot \nu \cdot \frac{dR}{d(x + \delta x)} \\ \&c. \&c.$$

But as no loss of generality necessarily results from the introduction of these auxiliary functions, the general integrals found by adding them to the primary functions will include the relative motions of any system of $(p+1)$ material points however distributed, and moving at a particular instant; and from the equal generality of the results, the real difficulties of the general integration of the equations (2) with selected primary functions will not be less than those which would be experienced in the integration of the original differential equations of motion. But if we have found *in any way* values of the functions $x_1, y_1, z_1, x_2, y_2, z_2$, &c., which represent approximately the motions in some particular case of mass distribution and motion with which we are concerned; and we can, therefore, *rightly* assume that in this particular case the ratios

$$\frac{\delta x}{r}, \frac{\delta y}{r}, \frac{\delta z}{r},$$

where

$$r = \sqrt{x^2 + y^2 + z^2}$$

are always small fractions; we can, in finding the auxiliary functions required for the complete integration of the differential equations of motion for the particular system with which we are concerned, restrict our investigations to those cases in which the ratios

$$\frac{\delta x}{r}, \frac{\delta y}{r}, \frac{\delta z}{r}$$

are small fractions; and the integration of the differential equations (2) can then be effected by successive approximations in which different orders of the fractions

$$\frac{\delta x}{r}, \frac{\delta y}{r}, \frac{\delta z}{r}$$

are neglected in succession.

We shall not obtain by following this method the general integrals for the relative motions of any system of $(p+1)$ material points, but those for the special system with which we are practically concerned.

If the primary functions x, y, z adopted for any moving mass, v , are definite functions of t , and contain no arbitrary constants to be found from observation, the six arbitrary constants required to give the necessary generality to the results must be introduced in the integration of the differential equations (2); and in this case the general integrals of the auxiliary functions $\delta x, \delta y, \delta z$ will have to be found.

But if the necessary constants have been introduced in the primary functions x, y, z , it will only be necessary to find particular integrals of the differential equations (2), and the work will be greatly facilitated in those cases in which the ratios

$$\frac{\delta x}{r}, \frac{\delta y}{r}, \frac{\delta z}{r}$$

are small fractions ; but in approximating to the expressions for these particular integrals it will be impossible to assign to the arbitrary constants contained in the functions x, y, z approximate numerical values, unless the errors which arise from the use of these approximate values are at every stage of the mathematical work of the order of the neglected terms

$$\frac{\delta x}{r}, \frac{\delta y}{r}, \frac{\delta z}{r}.$$

If we adopt, as is usual in the problems of gravitational astronomy, the complete integrals of the differential equations

$$(3) \quad \begin{aligned} \frac{d^2 x}{dt^2} &= -f(1 + \nu) \cdot \frac{x}{r^3}; \\ \frac{d^2 y}{dt^2} &= -f(1 + \nu) \cdot \frac{y}{r^3}; \\ \frac{d^2 z}{dt^2} &= -f(1 + \nu) \cdot \frac{z}{r^3} \end{aligned}$$

for the primary functions, we can express x, y, z under the forms

$$\begin{aligned} x &= r[\cos v \cdot \cos \Omega - \sin v \cdot \sin \Omega \cdot \cos \gamma] \\ y &= r[\cos v \cdot \sin \Omega + \sin v \cdot \cos \Omega \cdot \cos \gamma] \\ z &= r \cdot \sin v \cdot \sin \gamma \\ r &= a(1 - e \cos \phi) : \tan \frac{v - \omega}{2} = \sqrt{\frac{1 + e}{1 - e}} \cdot \tan \phi \\ \phi - e \sin \phi &= \sqrt{\frac{f(1 + \nu)}{a^3}} \cdot (t + \tau) \end{aligned}$$

where $a, e, \omega, \tau, \Omega, \gamma$ are six constants introduced in the integrations.

But the primary functions x, y, z thus obtained are not the integrals of the differential equations of motion unless the accelerating effects of the masses $\nu_1, \nu_2, \nu_3, \dots, \nu_p$ are insensible, and this is not the case in the problems with which we have to deal in astronomy. It will not, therefore, be possible to obtain with sensible accuracy the numerical values of the constants contained in the differential equations of motion and their integrals by equating the functions x, y, z with the coordinates of the moving masses as found from observation. If this is attempted, and definite values are assigned to x, y, z in the formation of the differential equations (2), the general integrals, instead of the partial integrals, of these equations will be required to ensure accuracy. If, however, we accept the primary elliptic integrals for the functions x, y, z , and use them, *in their general form*, in the formation of the differential equations

$$(4) \quad \begin{aligned} \frac{d^2 \delta x}{dt^2} &= -f(1 + \nu) \left[\frac{x + \delta r}{r^3} - \frac{x}{r^3} \right] + \int f \cdot \nu \cdot \frac{dR}{d(x + \xi)} \\ &\&c. = \&c. \end{aligned}$$

the *partial* integrals $\delta x, \delta y, \delta z$ found from these equations, free from any additional arbitrary constants, will by their addition to the primary functions x, y, z give solutions of the differential equations of motion. But the integration of the equations (4) will not lead to the required integrals of the differential equations of motion unless the functions x, y, z which are *actually used for the formation of the differential equations (2) are integrals of the primary differential equations (3)*. This point is of more practical importance than may at first sight appear to be the case, and it is liable to be overlooked in carrying out the laborious operations required for the determination of the auxiliary functions $\delta x, \delta y, \delta z$ for the different masses. The same units of length and of time will have to be adopted in the formation of the differential equations of motion for all the material points whose relative motions are under discussion ; or, what practically comes to the same thing, if different units are employed the ratios of the different units must be accurately known, and allowed for in the formation of the differential equations. But when the same units of length and of time are adopted, the numerical value of the factor f , which appears in each of the primary functions, will require that the same numerical value should be assigned to it in all the primary functions and retained throughout the investigations of the auxiliary functions $\delta x, \delta y, \delta z$. When, therefore, angular velocities

$$n_1, n_2, n_3 \dots n_p$$

are introduced to replace the complex functions

$$\sqrt{\frac{f(1+\nu_1)}{a_1^3}}, \sqrt{\frac{f(1+\nu_2)}{a_2^3}}, \sqrt{\frac{f(1+\nu_p)}{a_p^3}},$$

the primary differential equations will not be satisfied unless we have *identically for each material point*

$$(5) \dots f = \frac{a_1^3 n_1^2}{1+\nu_1}; f = \frac{a_2^3 n_2^2}{1+\nu_2}; f = \frac{a_p^3 n_p^2}{1+\nu_p};$$

and, in order that the units of length and of time may be the same throughout our mathematical investigations of the motion of the system, the same numerical value must be assigned to f in all the primary integrals, and we must have, therefore, the following identities :

$$(6) \dots \frac{a_1^3 n_1^2}{1+\nu_1} = \frac{a_2^3 n_2^2}{1+\nu_2} = \frac{a_p^3 n_p^2}{1+\nu_p}.$$

In the equations (6) $a_1, a_2, \dots$ and $n_1, n_2, \dots$ denote constants introduced in the integrations which have to be found from observations of the relative motions of the particular system under discussion, but they are not independent of each other. Indeed, if it were possible to overlook the conditional relations (6), instead of the $(7p+1)$ constants which are *sufficient* for the

complete solution of the physical problem under consideration, we should have placed at our disposal $8p$ constants determinable from observation.

These remarks are equally true when the method of the variation of parameters is employed. If a' , n' denote two variable parameters subject to the condition

$$(7) \quad f = \frac{n'^2 a'^3}{1 + \nu},$$

we shall have

$$(8) \quad 2a'^3 n' \cdot \frac{dn'}{dt} + 3n'^2 a'^2 \cdot \frac{da'}{dt} = 0,$$

or,

$$\frac{dn'}{dt} = -\frac{3}{2} \frac{n'}{a'} \cdot \frac{da'}{dt}$$

as a *necessary*, but not a *sufficient* conditional relationship between the parameters a' , n' for the solution of the physical problem under discussion. It will also be necessary for the solution of the problem of relative motions of the $(p+1)$ material points that the constants introduced in the integration of the differential equations from which a' and n' are found should be such that they give the same numerical value for f when substituted in the equation (7) for all the masses, ν_1 , ν_2 , &c.; and it is only by showing that this condition is satisfied that the truth of Newton's law of universal gravitation can, strictly speaking, be established.

The angular velocities

$$n_1 \cdot n_2 \dots \&c.$$

and the linear quantities

$$a_1 \cdot a_2 \dots \&c.$$

can, therefore, only be accurately found from observation subject to the conditions imposed by the identities (6), and the introduction of redundant constants into the theoretical expressions must be carefully avoided.

But if the expressions for the coordinates

$$x + \delta x, y + \delta y, z + \delta z,$$

which are compared with the results of observation, are *sensibly incomplete*, or *affected with sensibly erroneous empirical corrections*; and values of the angular velocities n_1 , n_2 , &c., and of the linear constants a_1 , a_2 , &c., or of their equivalents

$$\left(\frac{f(1 + \nu_1)}{n_1^2} \right)^{\frac{1}{3}}, \left(\frac{f(1 + \nu_2)}{n_2^2} \right)^{\frac{1}{3}}, \&c.,$$

are thus obtained from a discussion of a limited series of residual discordances by the method of least squares or otherwise, there is no mathematical necessity whatever, if Newton's law of gravitation is true, that the equations of condition (6) should be satisfied by the values of the constants thus found ; or any proof that the physical problem has been solved by the process followed, unless it can be shown that the values of f , found from the discussion of the residual discordances, in connection with the constants $n_1, n_2, \&c., \nu_1, \nu_2, \&c.$, are sensibly the same for all the material points.

But in the problems of the relative heliocentric motions of the centres of gravity of the planets, the motions under mathematical treatment are not directly dependent upon the rotation of the Earth about its axis, or on the linear dimensions of the Earth ; and it is possible in the discussions of these relative motions to clear the fundamental constant f from fallible errors of determination, and to render its value *absolutely definite throughout our mathematical investigations by the selection of suitable units of length and of time*. But when this course is followed, the angular velocity of the Earth about its axis and the linear dimensions of the Earth referred to the units selected to render f definite will have to be found from a discussion of observations, and the values which will be found from these discussions will necessarily be liable to fallible errors of determination.

The difficulties thus presented are inherent in the physical problem under consideration, and they cannot be avoided by any mathematical artifices of change of constants or of variables. The close agreement between the values recently obtained for the constant of solar parallax from discussions of such very different geocentric motions as those of the Moon, Venus, Mars, and the asteroids, values which may be said to be confined within such limits as $8''.80$ and $8''.90$, is sufficient to prove that the outstanding errors of our geocentric coordinates are confined within small limits. But the exact values of the solar parallax cannot be obtained with certainty from such discussions, however accurately the geocentric distances of a planet from neighbouring stars or from the Sun are measured at any particular time, until the ratios of the linear constants

$$a_1, a_2, \&c.,$$

as well as the values of such constants as

$$e_1, e_2, \&c.,$$

are accurately known. I should be inclined to believe that the progress of gravitational astronomy is at present retarded from incompleteness of our theoretical investigations, the employment of inconsistent conditions between the variable t and

facts of observation, and the introduction of redundant constants to diminish the residual errors formed from observations extending over only limited intervals of time, rather than from existing errors of observation. But such errors are undoubtedly quite sensible, and every step which tends to diminish them is one in the right direction, and should be most carefully utilised.

Note on the Distribution of the Stars generally used for the Determination of Clock-Error. By E. J. Stone, M.A., F.R.S., Radcliffe Observer.

The authorities at Greenwich for a great many years have annually published and distributed a list of the Mean Right Ascensions of some 215 stars, brought up from the latest *Greenwich Catalogue*, and corrected, when necessary, from recent observations. This list has been found exceedingly useful by many astronomers, more particularly for the determination of the errors of their clocks. The apparent places of most of the Greenwich stars are included in the current *Nautical Almanacs*; and I understand that the whole of them will be included in future volumes. But the *distribution* of the clock-stars in this list about the equator is not very satisfactory, and I have thought that this might easily be improved; and, to test opinion on this point, the following list has been prepared. I have not ventured to cut out any of the stars which are now on the Greenwich list; but if the number of stars required to fill up the gaps should be thought to render the total number of stars inconveniently large for that constant re-observation which is desirable, there are some of the stars now in the list which might be cut out without any serious practical loss. The Right Ascensions given for the additional stars are close approximations to the true values; and although they are not at present as accurate as the Right Ascensions of most of the other stars, the errors are so small that, if adopted for clock-stars, their places would be rapidly improved without leading to any practical inconvenience.

Two diagrams showing the general distribution of the stars are appended.